Fourier Transform and Spectral Analysis

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Introduction

- Historical Background
- Fourier Synthesis
- Fourier Analysis
- Time Frequency Analysis
- Applications

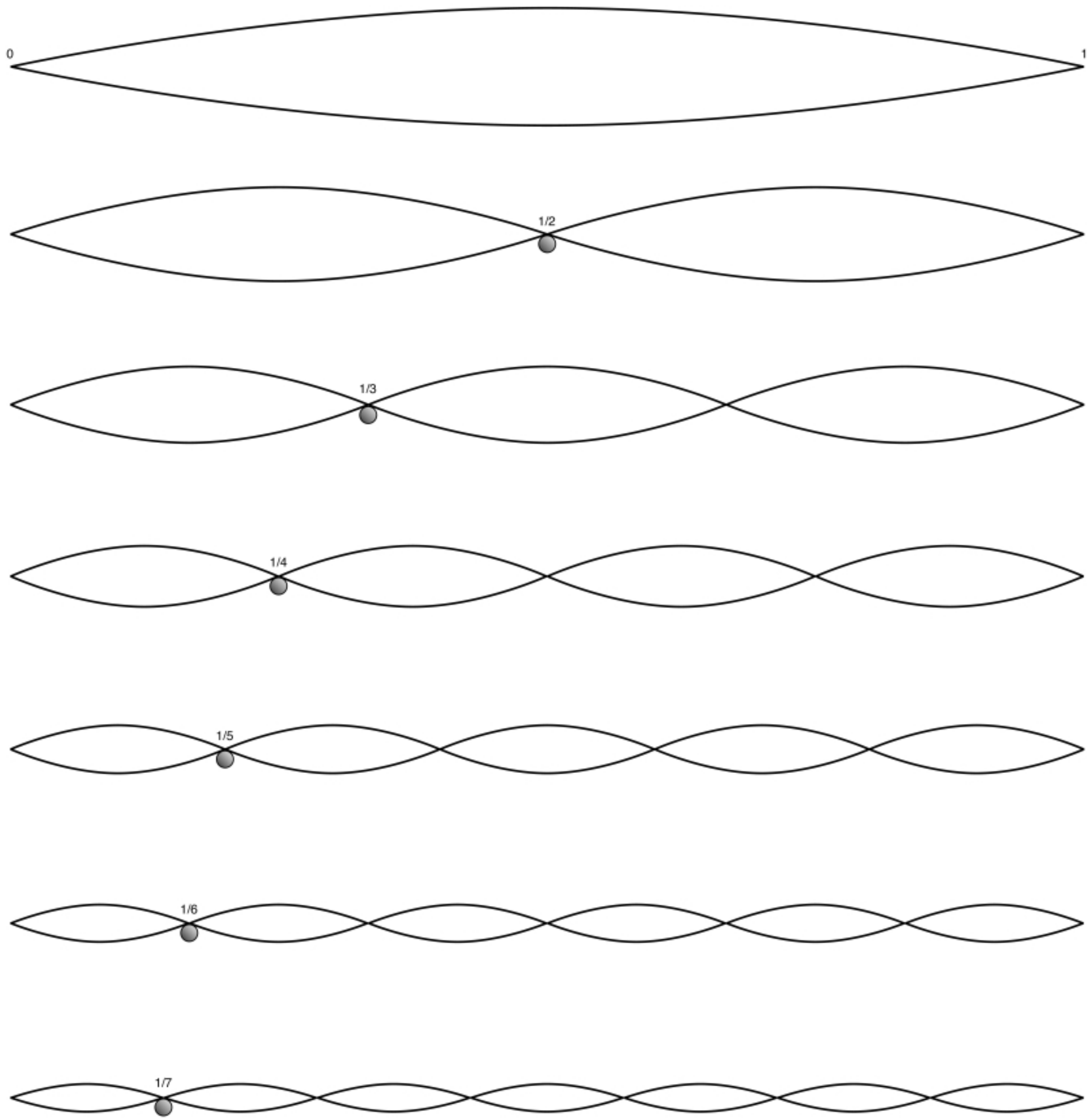
Jean Baptiste Joseph Fourier (1768-1830)

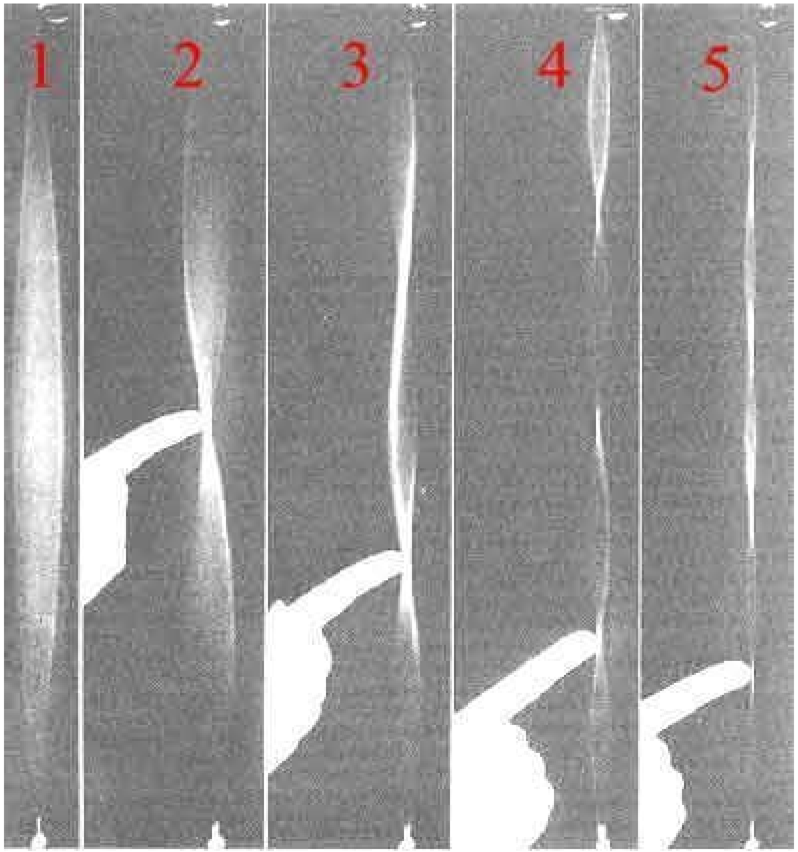
Théorie analytique de la chaleur (1822)

A continuous periodic function can be written as a sum of sine-waves

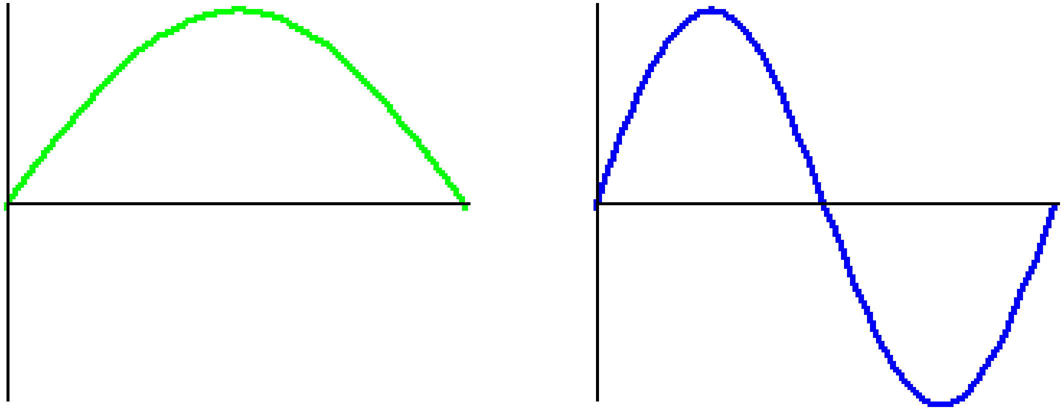
$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\omega t) - b_n \sin(n\omega t)).$$

Vibrating Strings





Vibrating Strings



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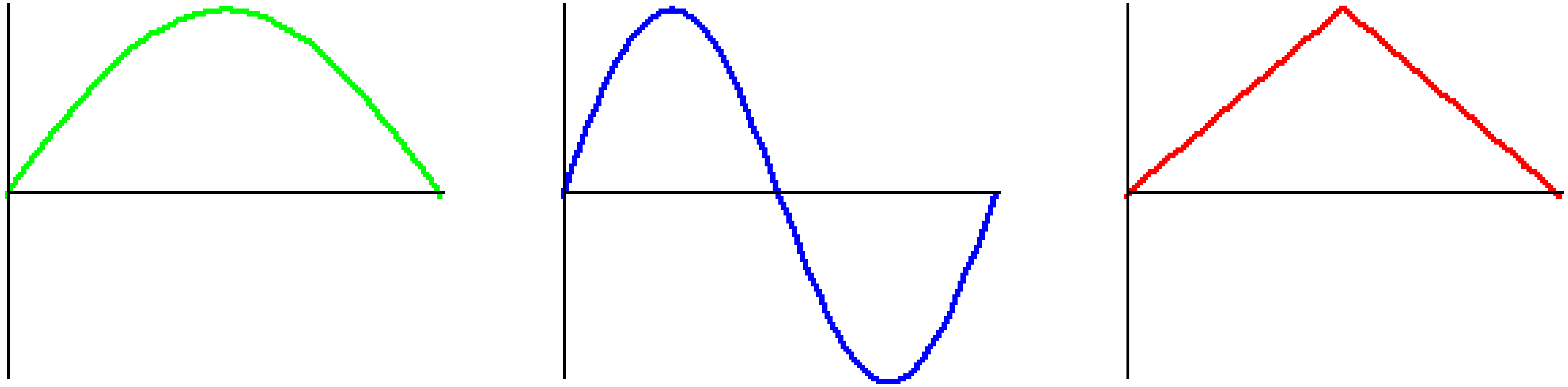
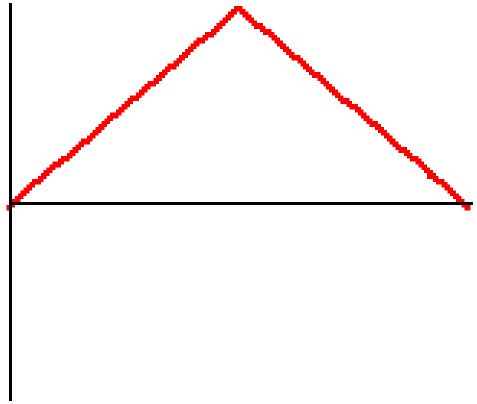
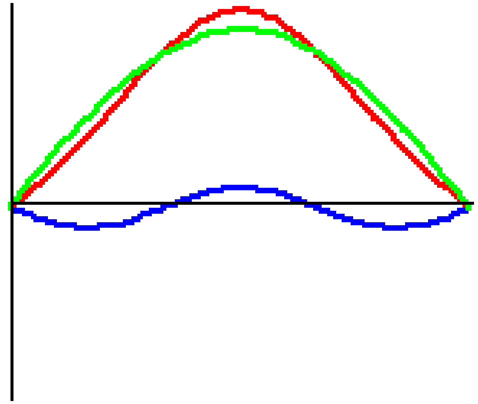
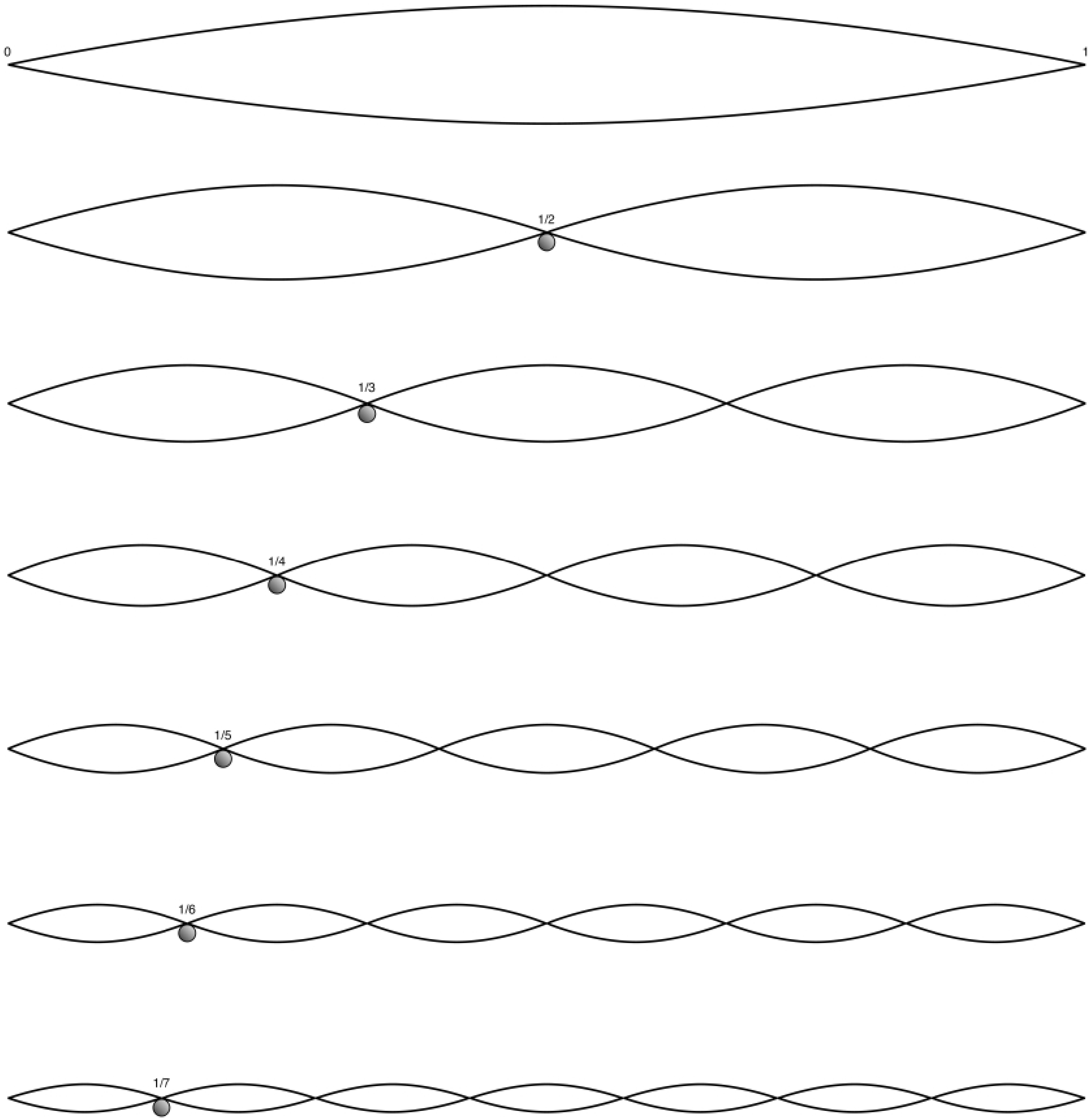


Vibrating Strings







a_1 ω

a_2 2ω

a_3 3ω

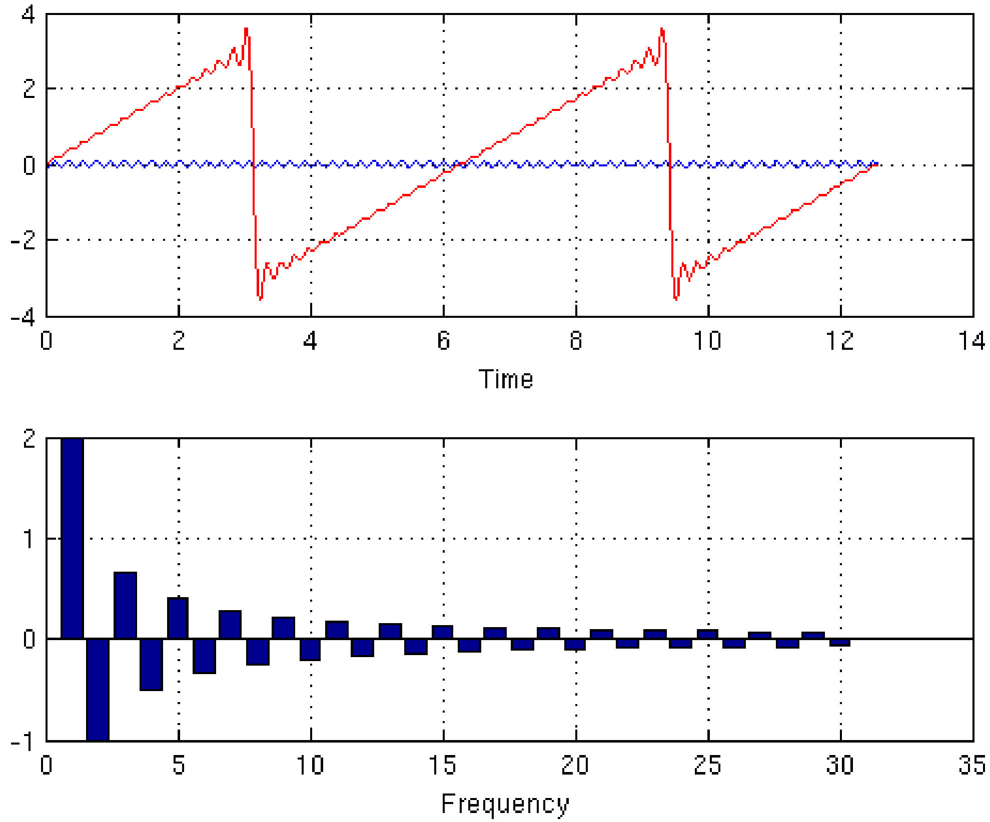
a_4 4ω

a_5 5ω

a_6 6ω

a_7 7ω

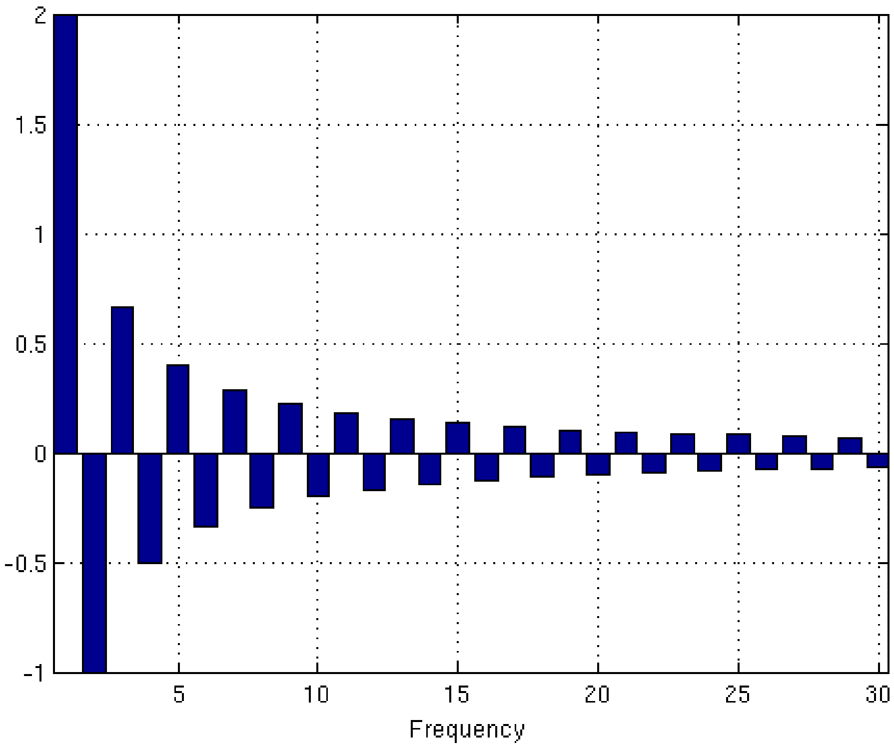
Sawtooth



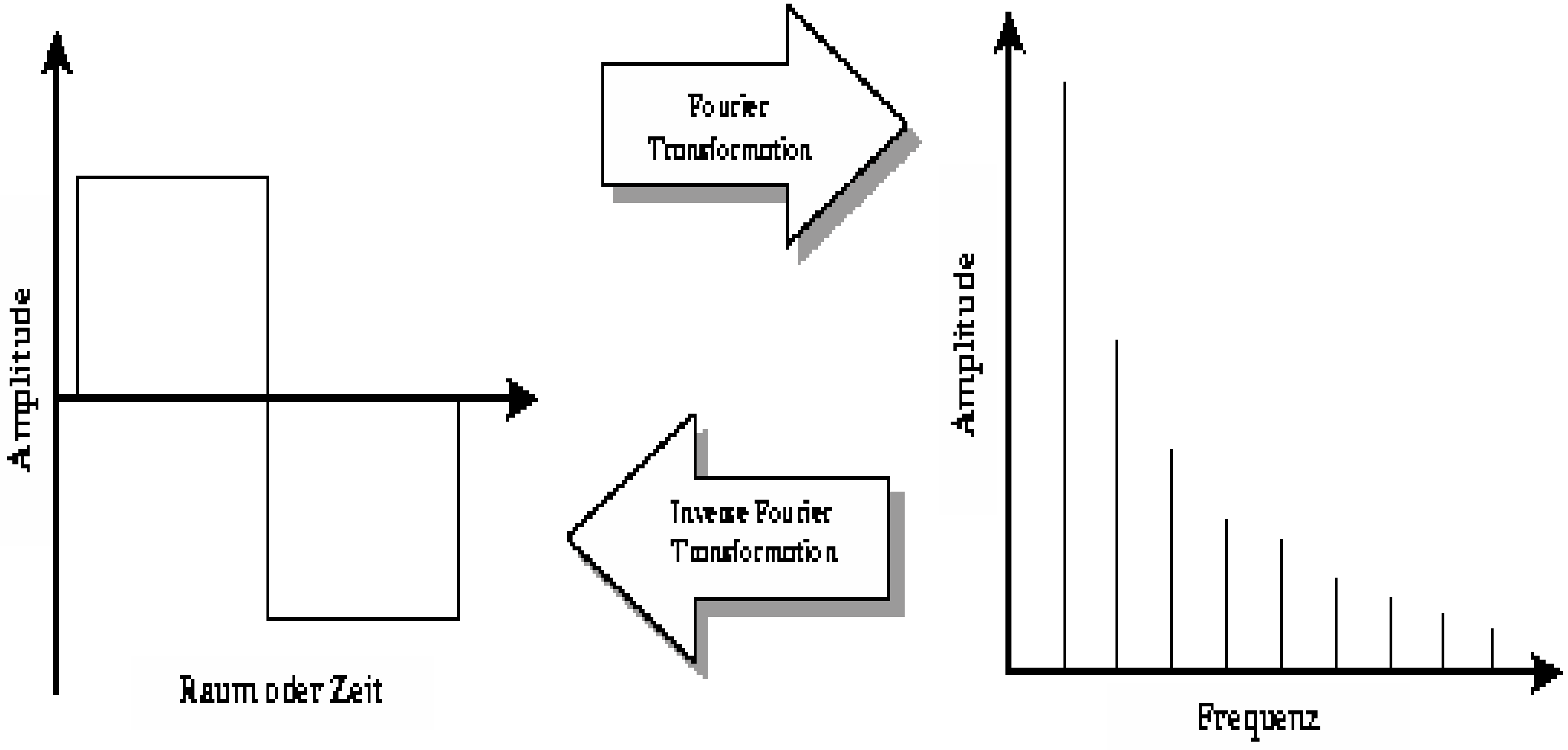
$$c(t) = 2\sum_{k=1} \frac{(-1)^k}{k} sin(k \cdot t)$$

A continuous periodic function can be written as a sum of sine-waves

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\omega t) - b_n \sin(n\omega t)).$$



Coefficients are forming the spectrum



$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\omega t) - b_n \sin(n\omega t)).$$

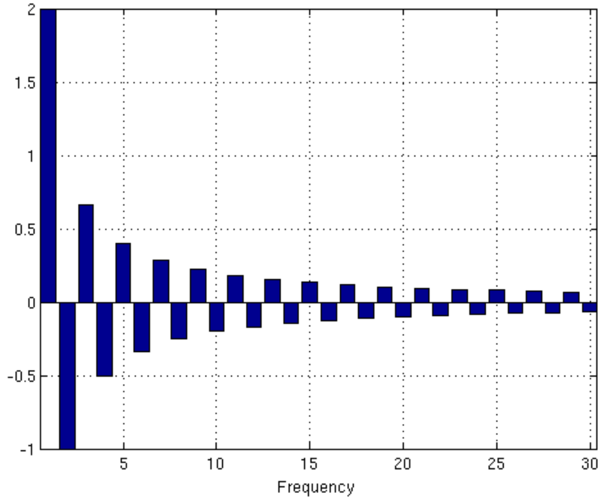
Fourier series and complex numbers

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\omega t) - b_n \sin(n\omega t)).$$

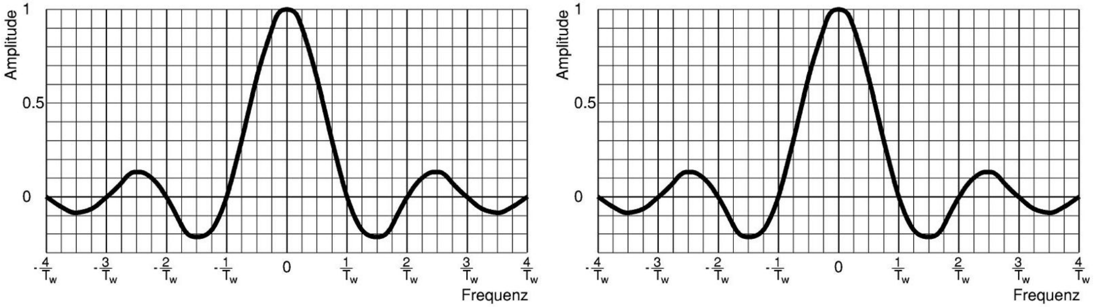
$$= a_0 + \sum_{n=1}^{N} \frac{1}{2} \left(a_n(e^{\mathrm{i}n\omega t} + e^{-\mathrm{i}n\omega t}) + \mathrm{i}b_n(e^{\mathrm{i}n\omega t} - e^{-\mathrm{i}n\omega t})\right)$$

$$f(t) = \sum_{n=-N}^{N} c_n e^{\mathrm{i}n\omega t}$$

From periodic signals to continuous functions



$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\omega t) - b_n \sin(n\omega t)).$$



$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(\omega) e^{i\omega t} \, d\omega$$

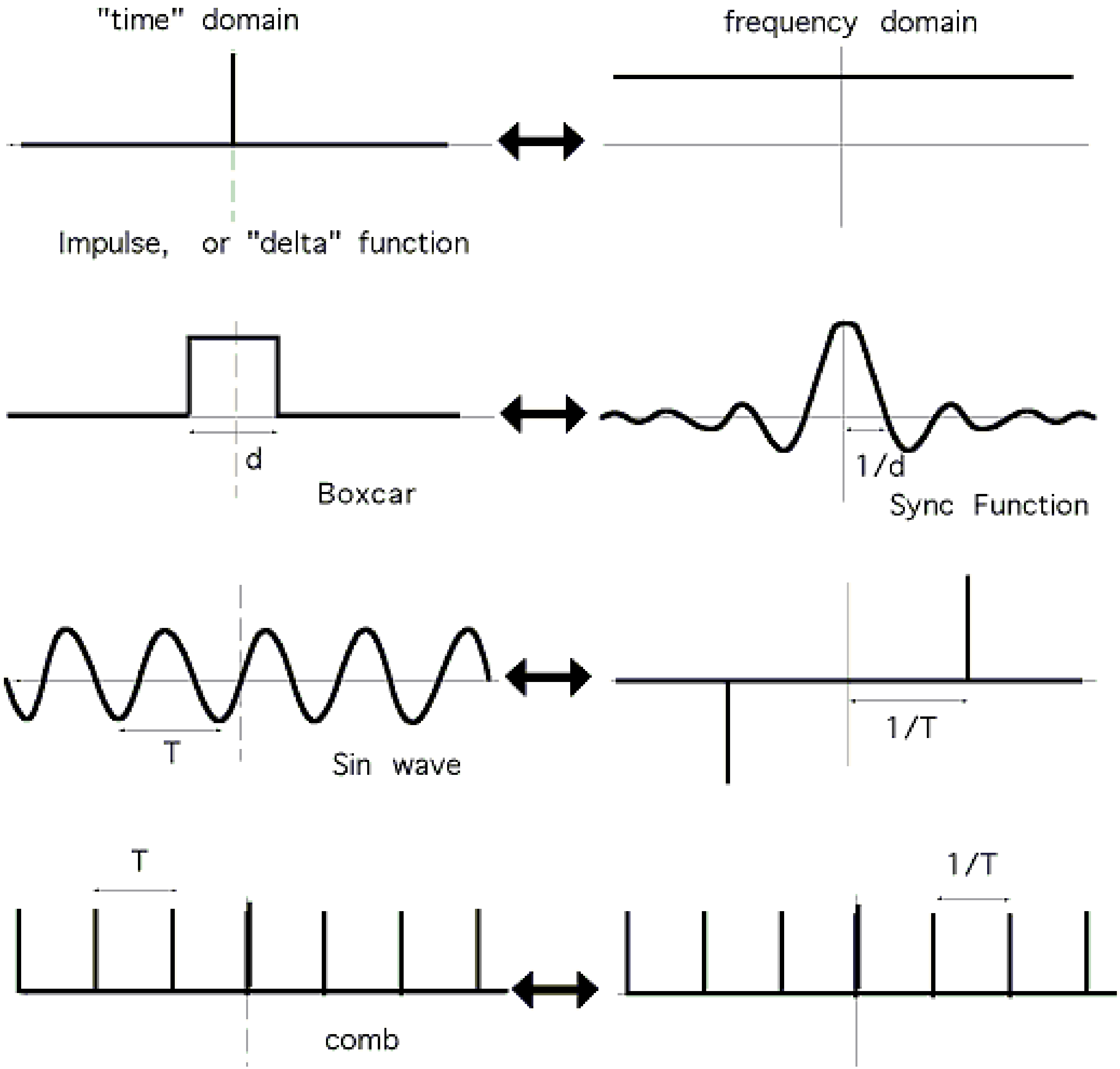
Spectrum has a real and an imaginary part !

From periodic signals to continuous functions

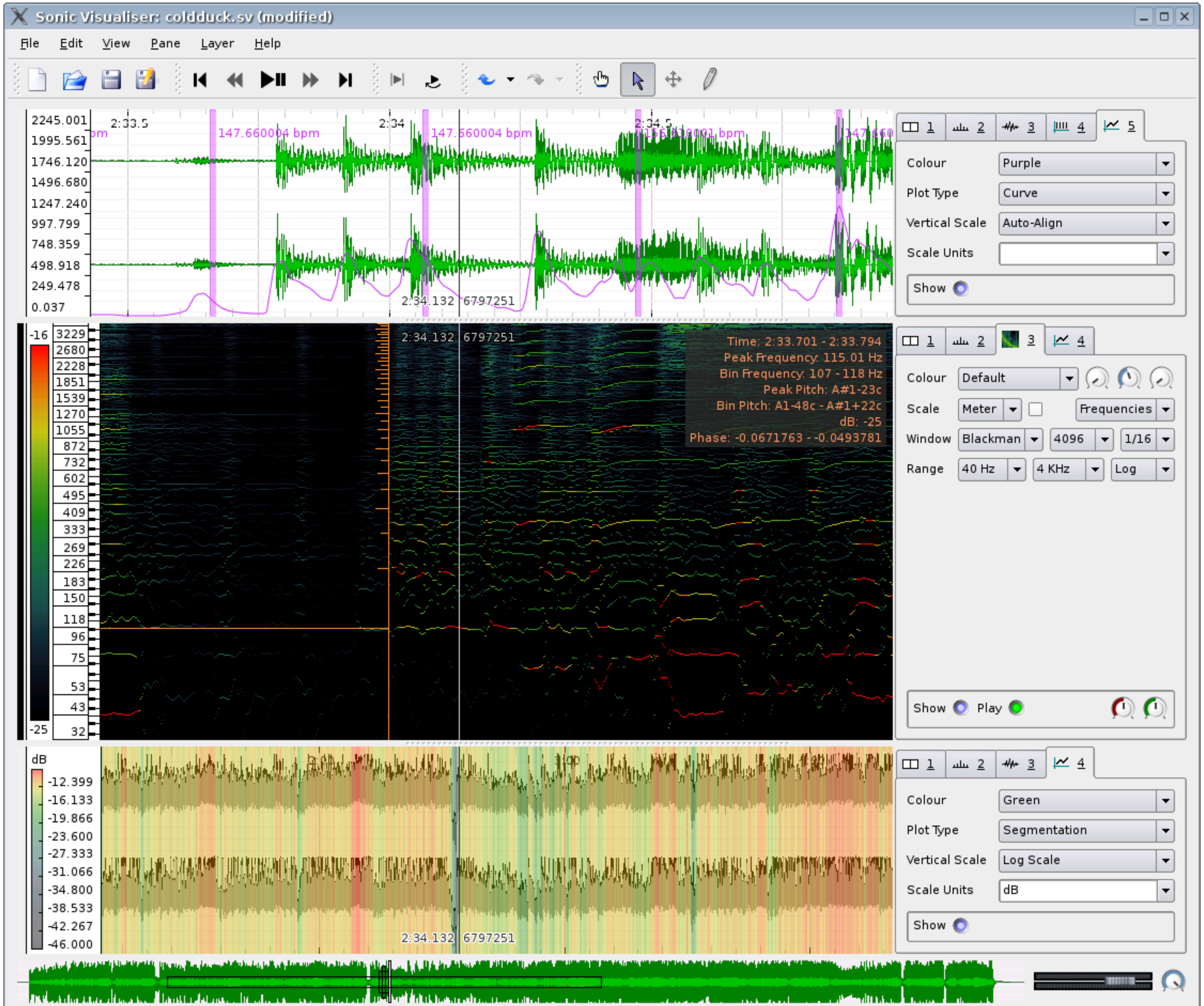
$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(\omega) e^{\mathrm{i}\omega t}\, d\omega$$

$$a(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i}\omega t}\, dt.$$

Spectral Density: $\Phi(\omega) = a(\omega) \cdot \overline{a(\omega)}$


"time" domain
frequency domain
Impulse, or "delta" function
d
Boxcar
1/d
Sync Function
T
Sin wave
1/T
T
1/T
comb

Typical Applications: Signal Analysis



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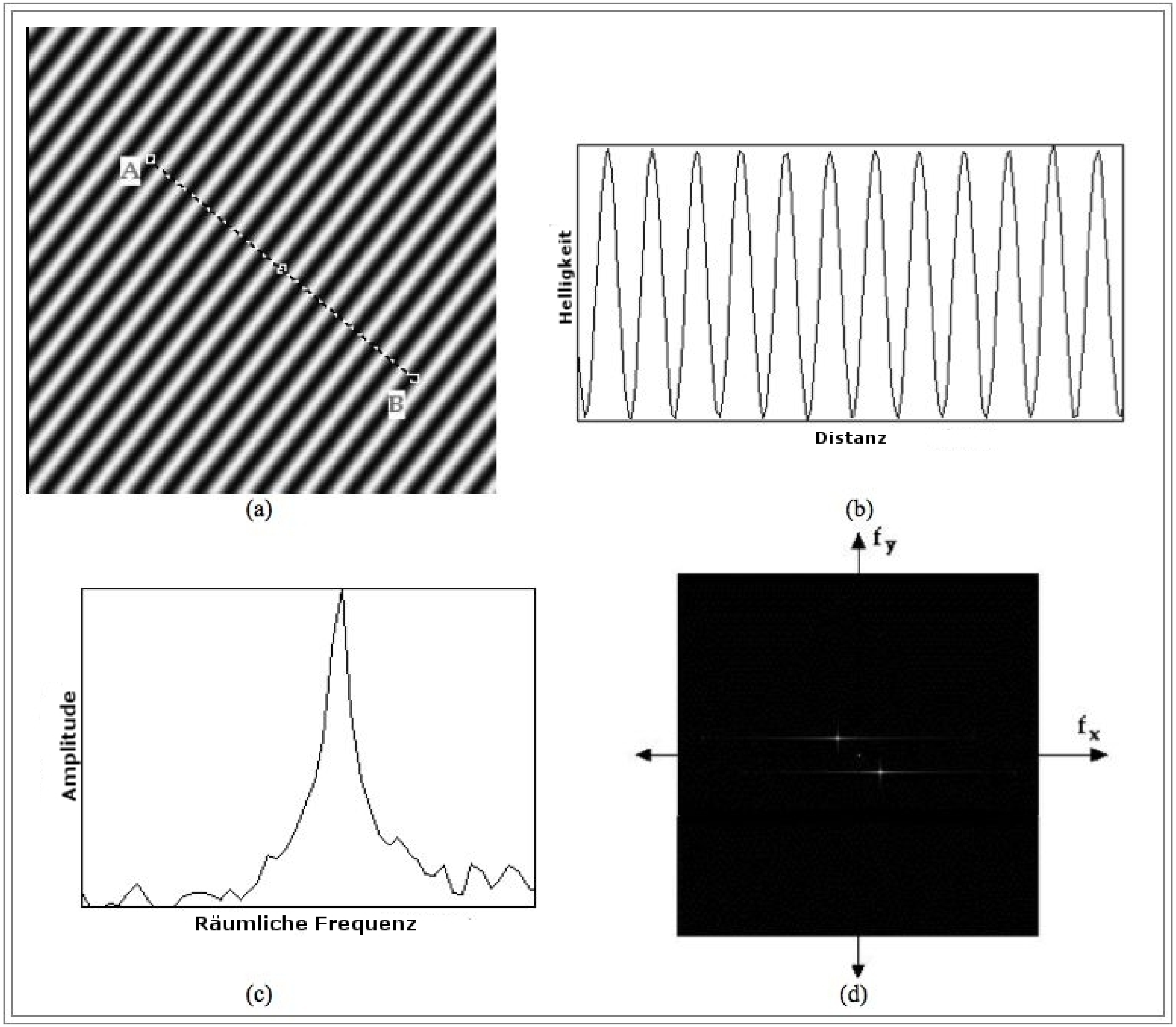
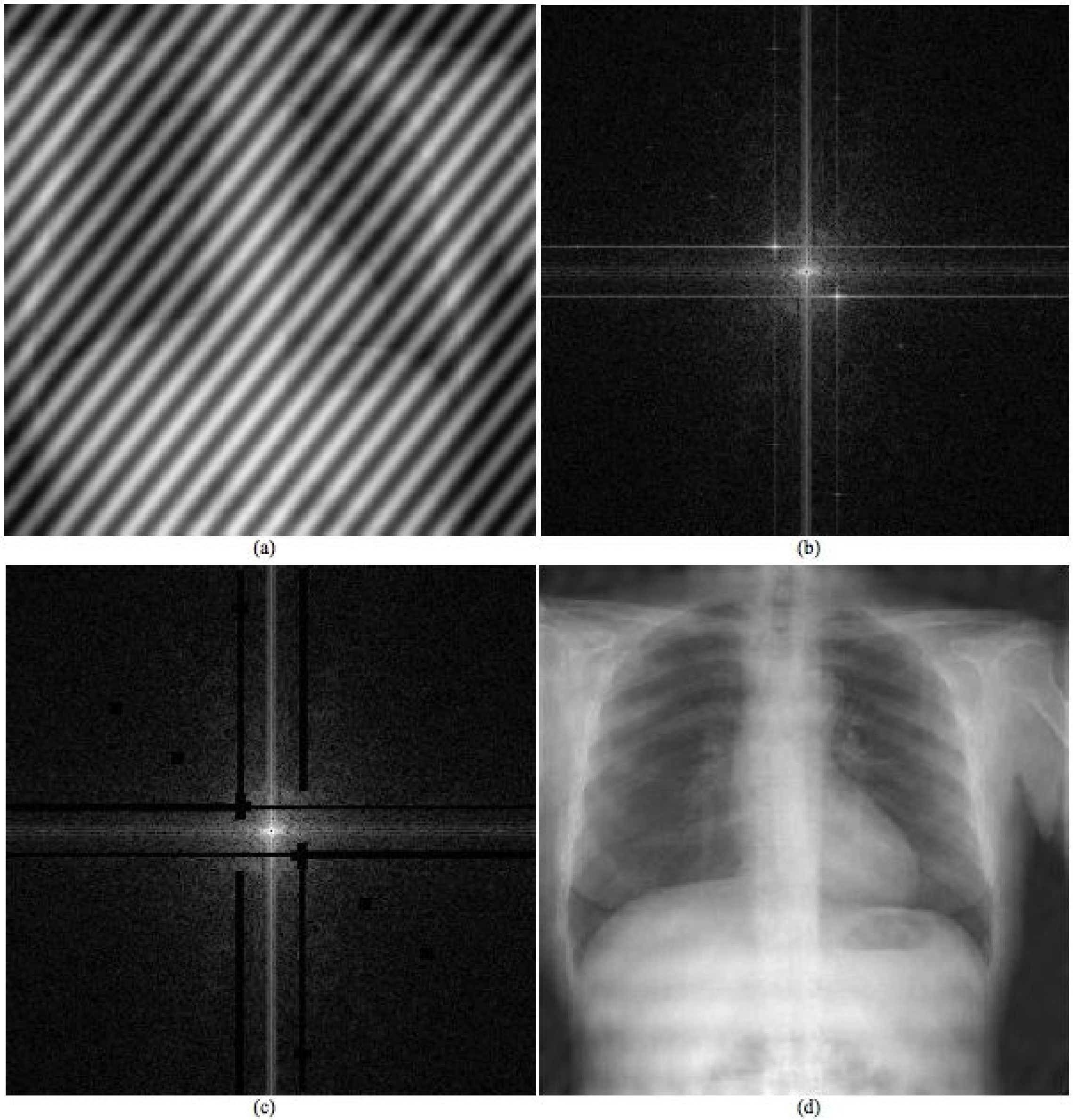
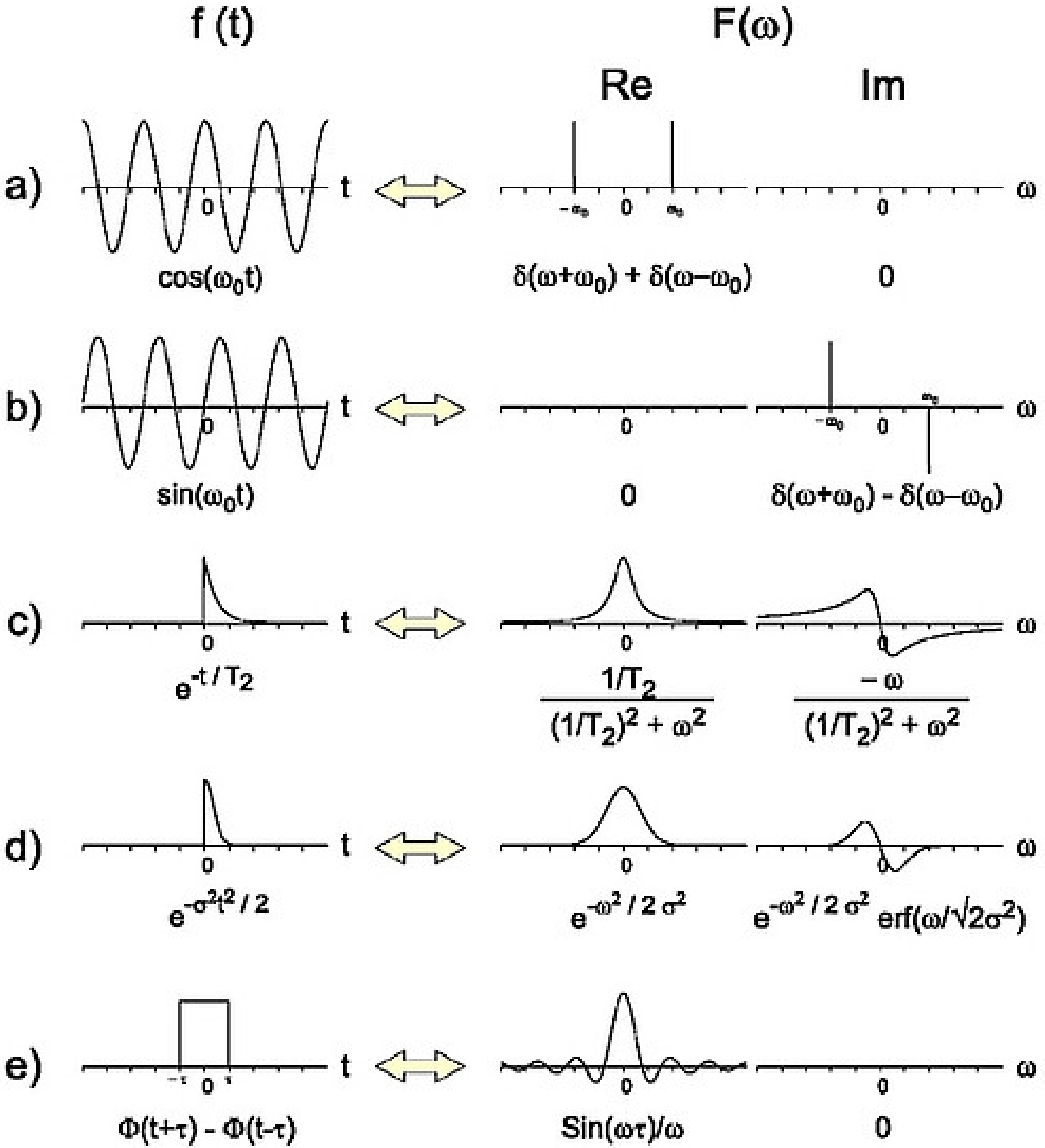
Typical Applications: Image Processing

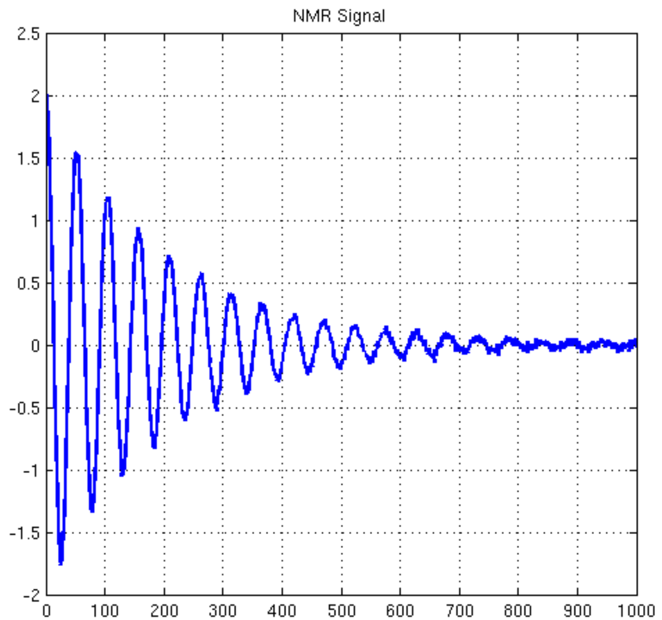


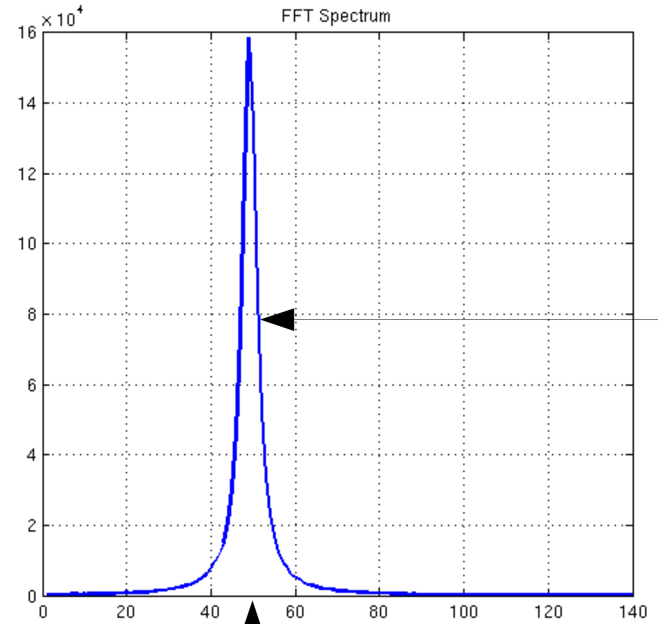
Typical Applications: Image Processing




f (t)
F(ω)
Re
Im
a)
cos(ω0t)
δ(ω+ω0) + δ(ω−ω0)
0
b)
sin(ω0t)
0
δ(ω+ω0) - δ(ω−ω0)
c)
e-t / T2
1/T2 / (1/T2)2 + ω2
− ω / (1/T2)2 + ω2
d)
e-σ2t2 / 2
e-ω2 / 2 σ2
e-ω2 / 2 σ2 erf(ω/√2σ2)
e)
Φ(t+τ) - Φ(t-τ)
Sin(ωτ)/ω
0

Typical Applications: Signal Analysis





$$\frac{1}{\pi T_2}$$

$$\omega$$

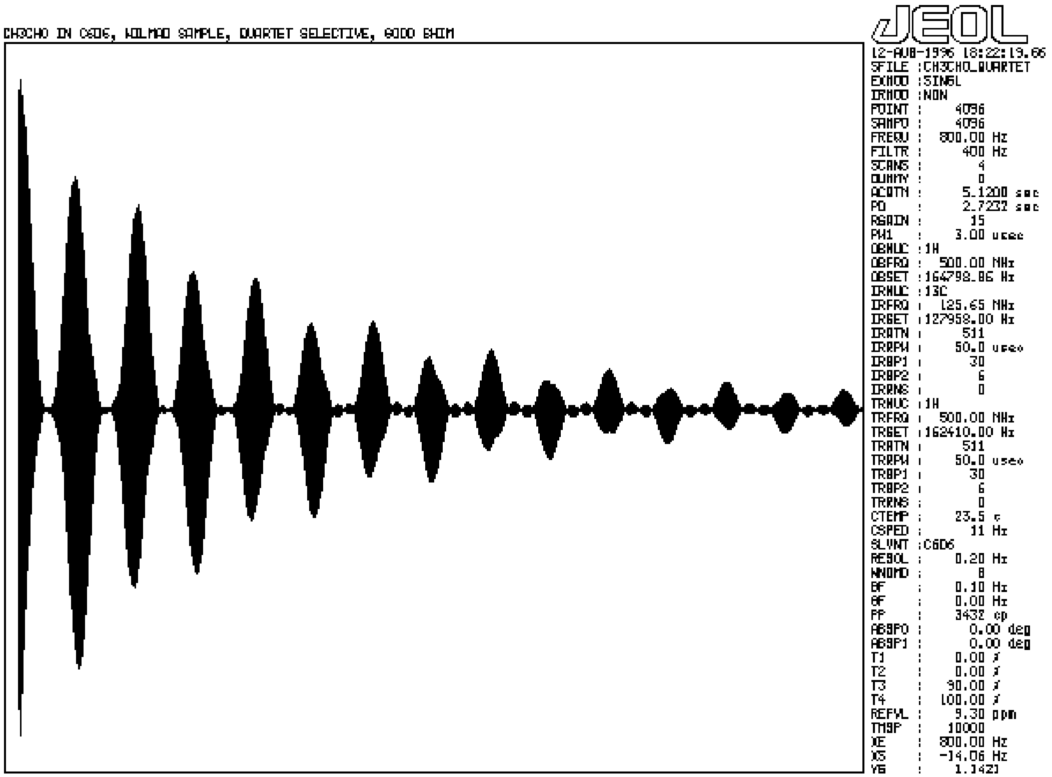
$$f(t) = e^{i\phi} e^{i\omega t} e^{\frac{t}{T_2}}$$

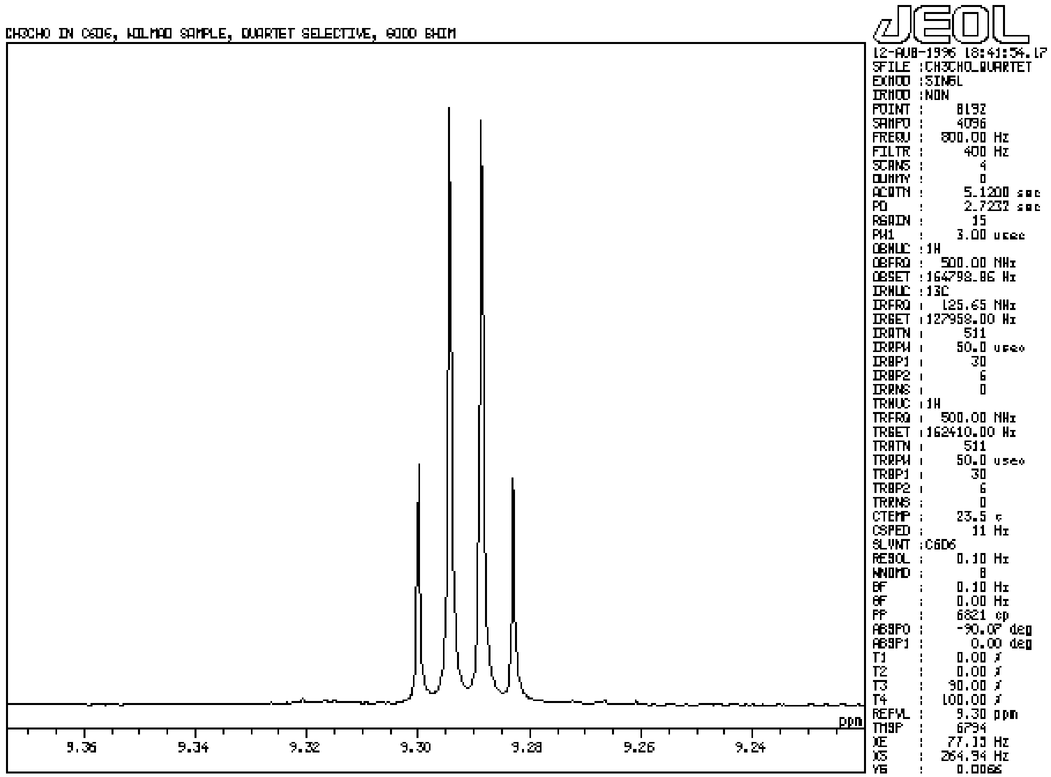
phase

oscillation

decay

The Sound of Chemistry

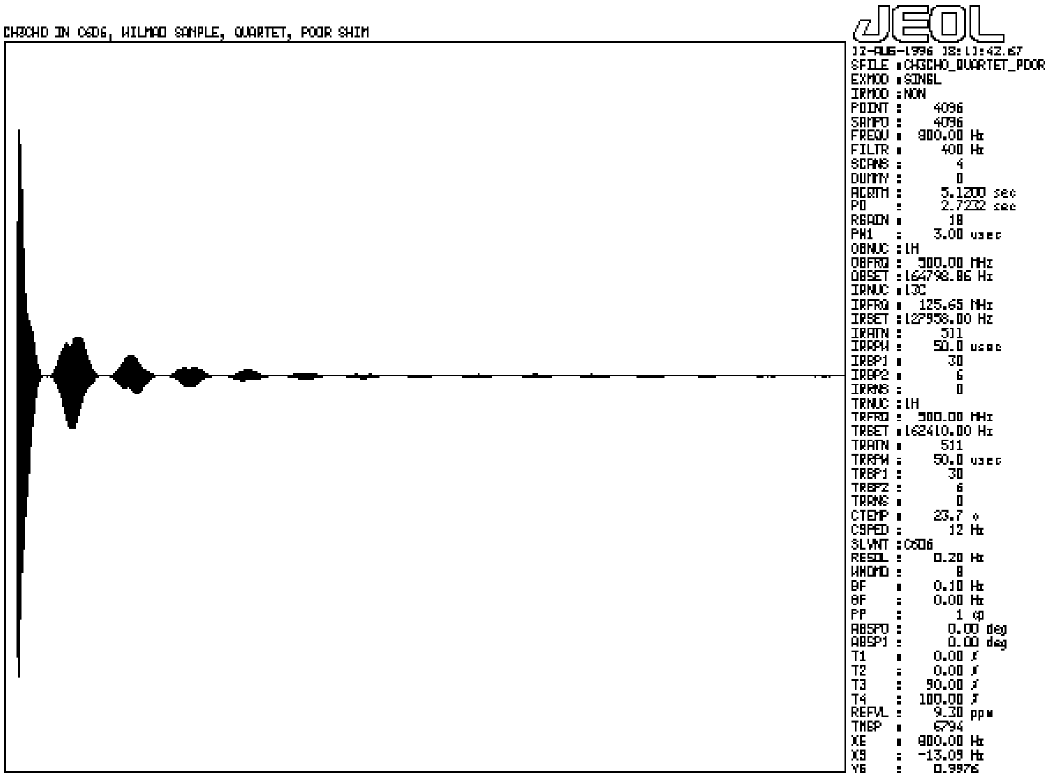


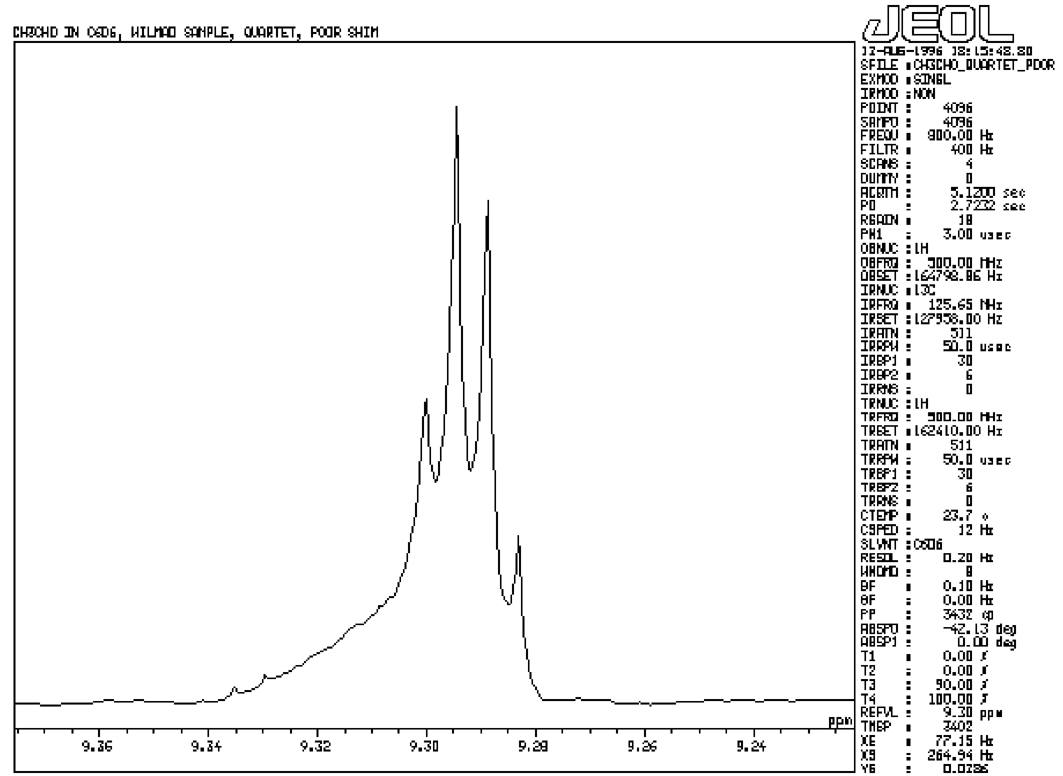


CH3CHO FID

CH3CHO Spectrum

The Sound of Chemistry





CH3CHO FID

CH3CHO Spectrum

That's it!

References

- http://de.wikipedia.org/wiki/Fourier-Transformation
- http://en.wikipedia.org/wiki/Fourier_transform
- http://www.chemie.uni-erlangen.de/oc/research/NMR/music.html
- http://www.sonicvisualiser.org